Chapter 4
Polynomials

Abstract In this section we will work on some examples that will cover the section on polynomials required in SMR 1.2 of the CSET exam. We will show how you can find the roots of a polynomial using theorems such as the factor theorem, rational root theorem and conjugate root theorem as well as how to graph polynomials.

What is a polynomial?
A polynomial is an algebraic expression of powers of x of the type

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where n is a whole number and the coefficients $a_n,\ a_{n-1}, \cdots, a_2,\ a_1,\ a_0$ are any real numbers. For example,

$$-2,\ x,\ 9x^5 - 1,\ 3x^4 + 5x^3 - 2x^2 - 10x + 15,\ 7x^{10} + 5x - 9$$

are all polynomials. However, the following are not polynomials:

$$x^{-2},\ -2x^{2/3},\ \frac{1}{x^2 - 3x + 4},\ \frac{3x-2}{x^2 - 3x + 4},\ 2^x - 3$$

The highest power of x in a polynomial $f(x)$ is called the degree of $f(x)$, and the term of this highest power is called the leading term of $f(x)$. A polynomial consisting of exactly one term is call monomial, a polynomial with exactly two terms is called binomial, and a polynomial with exactly three terms is called trinomial.

In this chapter we will discuss the following regarding polynomials:
(a) How to find roots of a polynomial.
(b) How to apply theorems about the behavior of roots, and factors of polynomials.
(c) How to factor a polynomial expression.
(d) How to sketch the graph of a polynomial.
(e) Given a graph, how to find, and analyze, the polynomial with that graph.

4.1 Finding Roots of Polynomials

A root of a polynomial $f(x)$ (also called a zero of $f(x)$) is a number that, when plugged into $f(x)$, yields zero. In other words, a root of $f(x)$ is a number α such that $f(\alpha) = 0$.

Example 4.1. We want to show that 1 is a root of the polynomial $f(x) = x^3 + 3x^2 - x - 3$.
Substitute 1 for x in $f(x)$ to get

$$f(1) = 1^3 + 3 \cdot 1^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0$$

Since $f(1) = 0$ then 1 is a root of $f(x)$.

In the previous example we see that it is easy to check whether a given number is a root of a given polynomial. The more interesting question is: Given a polynomial, how do we find its roots?

First of all, we need to answer a simpler question: How many roots does a polynomial have? The answer is simple: A polynomial of degree n has exactly n roots.

Note: Not all these roots need to be integers or rational numbers; They can be irrational or complex. Also, there can be multiple (repeated) roots. For instance, look at the following table:

Polynomial	Roots
x^2-3x+2	1, 2
x^2-4x+4	2, 2 (multiple root)
x^3-6x^2+13x	0, $3+2i$, $3-2i$ (one real and two complex roots)
x^3-x^2-2x+2	1, $\sqrt{2}$, $-\sqrt{2}$ (one rational and two irrational roots)

You can clearly see, in the examples above, that the nature of the roots vary but the number of roots is exactly equal to the degree of the polynomial.
The big question is: How do we find these roots?
There are a couple of very useful theorems that we can use:

Theorem 4.1 (Rational Root Theorem). *Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial with integer coefficients. Assume that $\frac{p}{q}$ is a rational root of $f(x)$ (where p and q have no common factors), then*
(a) p divides a_0, and
(b) q divides a_n.

You will see how this theorem is used in the worked out examples at the end of this section.

Theorem 4.2 (Factor Theorem). *A polynomial $f(x)$ has $(x-c)$ as a factor if and only if c is a root of $f(x)$.*

These theorems can be used together to factor a polynomial with integer coefficients, by allowing the rational root theorem to get a root and then using the factor theorem to find a factor of $f(x)$.

Example 4.2. Remember that 1 was a root of the polynomial $f(x)=x^3+3x^2-x-3$ (from example 4.1). Hence, using the factor theorem we find that $(x-1)$ is a factor of the polynomial, which is true because $f(x)$ can be factored as (using techniques that we will learn shortly):

$$x^3+3x^2-x-3=(x-1)(x+3)(x+1)$$

Finding Roots of Polynomials: Worked out examples

1. *Find the roots of $f(x)=x^3-2x^2-x+2$. Also, factor this polynomial.*
Answer: Since $f(x)$ is a polynomial with integer coefficients, trying to use the rational root theorem is a good idea. So, if $\frac{p}{q}$ were a rational root of $f(x)$ then q must divide the leading coefficient, which is a 1. It follows that $q=\pm1$. Also, p must divide the constant term, that is p divides 2, and thus $p=\pm1,\pm2$. Hence, taking into account all possibilities for p and q, we get that the candidates to be roots of $f(x)$, given by the rational root theorem, are $-2,-1,1,2$.

If we plug in $x=-2$, we get $f(x)=(-2)^3-2(-2)^2-(-2)+2=-12\neq0$
If we plug in $x=-1$, we get $f(x)=(-1)^3-2(-1)^2-(-1)+2=0$
If we plug in $x=1$, we get $f(x)=(1)^3-2(1)^2-(1)+2=0$
If we plug in $x=2$, we get $f(x)=(2)^3-2(2)^2-(2)+2=0$

It follows that $-1,1$ and 2 are roots of $f(x)$. Because this polynomial has a degree of three and we found three roots, we know we have found *all* the roots of $f(x)$.
Now, to factor $f(x)$ we use the factor theorem. This theorem says that $(x-(-1)),(x-1)$ and $(x-2)$ are factors of $f(x)$. It follows that

$$f(x)=(x+1)(x-1)(x-2)$$

Finally, by multiplying the factors on the right, we can check that the factoring found is correct.

2. *Find the roots of $f(x) = x^3 - x^2 - 2x + 2$. Also, factor this polynomial.*
Answer: As above, we will first look for roots using the rational root theorem, which tells us that the only candidates to be rational roots of $f(x)$ are $-2, -1, 1, 2$.

If we plug in $x = -2$, we get $f(x) = (-2)^3 - (-2)^2 - 2(-2) + 2 = -6 \neq 0$
If we plug in $x = -1$, we get $f(x) = (-1)^3 - (-1)^2 - 2(-1) + 2 = 2 \neq 0$
If we plug in $x = 1$, we get $f(x) = (1)^3 - (1)^2 - 2(1) + 2 = 0$
If we plug in $x = 2$, we get $f(x) = (2)^3 - (2)^2 - 2(2) + 2 = 2 \neq 0$

It follows that 1 is the only rational root of $f(x)$. Since this polynomial has a degree of three, we still need to find two more roots. Clearly these two unknown roots cannot be rational numbers. So these roots have to be either irrational or non-real. How do we find these roots?
First, since 1 is a root, by the factor theorem $(x - 1)$ is a factor. Since the polynomial is of degree 3, the remaining factor must be quadratic; we can use synthetic division to find the factor.

$$\begin{array}{c|cccc} 1 & 1 & -1 & -2 & 2 \\ & & 1 & 0 & -2 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

The first three entries of the last row: $1, 0$ and -2 give the coefficients of $1 \cdot x^2 + 0 \cdot x - 2 = x^2 - 2$. It follows that

$$f(x) = (x-1)(x^2-2)$$

Since the solutions of $x^2 - 2 = 0$ are $\pm\sqrt{2}$, the three roots of $f(x)$ are $-\sqrt{2}, 1, \sqrt{2}$.
Now we can factor $f(x)$ completely as

$$f(x) = (x-1)(x-\sqrt{2})(x+\sqrt{2})$$

Finally, by multiplying the factors on the right, we can check that the factoring found is correct.
3. *Find the roots of $f(x) = x^3 - x^2 + 2$.*
Answer: We use the rational root theorem, and we find that the possibilities for rational roots are $-2, -1, 1, 2$. When plugging these numbers into $f(x)$, we get that -1 is the only rational root of $f(x)$. We need to search for the other roots, so we proceed as in the previous example.
Being that -1 is a root of $f(x)$, we know that $(x + 1)$ is a factor of $f(x)$. Moreover, dividing yields

$$f(x) = (x+1)(x^2 - 2x + 2)$$

In order to get the roots of $g(x) = x^2 - 2x + 2$, we use the quadratic formula and get

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

It follows that $1 - i$ and 1_i are the two missing roots of $f(x)$. Hence, all the roots of $f(x)$ are -1, $1 - i$ and $1 + i$.
4. *Find the roots of $f(x) = 2x^3 - 7x^2 + 4x + 3$.*
Answer: As we have done before, using the rational root theorem, we get quite a few candidates for the rational roots of this polynomial:

$$-3,\ -2,\ -\frac{3}{2},\ -1,\ -\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ 3$$

By plugging these numbers into $f(x)$ we see that $\frac{3}{2}$ is a root. What are the other roots?
Using synthetic division, as before, we see that the other quadratic factor is $2x^2 - 4x - 2$. Using the quadratic

equation, we can solve this to yield the remaining two roots, which are $1-\sqrt{2}$ and $1+\sqrt{2}$. Hence, the roots of $f(x)$ are $\frac{3}{2}$, $1-\sqrt{2}$ and $1+\sqrt{2}$.

5. *Find the roots of the polynomial* $f(x)=2x^4-7x^3+5x^2+x-1$.
Answer: This is a degree 4 polynomial, but the approach is nevertheless the same.
Using the rational root theorem, we see that the possible rational roots of $f(x)$ are

$$-1,\ -\frac{1}{2},\ \frac{1}{2},\ 1$$

When you plug in 1, you see that it is a root of $f(x)$ (you could also see that $\frac{1}{2}$ is a root, but for the time being we will just concentrate on 1 as a root).
Since 1 is a root, then $(x-1)$ is a factor of $f(x)$. Thus, we use synthetic division to factor out $(x-1)$. Since $f(x)$ has degree 4, the division should yield a factor of degree 3. In fact, we get $2x^3-5x^2+1$. Hence

$$f(x)=(x-1)(2x^3-5x^2+1)$$

We need to find the roots of that cubic polynomial. Similar to previous problems, we use the rational root theorem and synthetic division to find that the roots of the cubic polynomial are $\frac{1}{2},\ 1+\sqrt{2},\ 1-\sqrt{2}$.
We have found the four roots of $f(x)$. They are $1,\ \frac{1}{2},\ 1+\sqrt{2},\ 1-\sqrt{2}$.

6. *Find the roots of the polynomial* $f(x)=x^4-4x^3+7x^2-6x+2$.
Answer: Using the rational root theorem, we see that the possible rational roots are $-2,-1,1$ and 2. Out of these, (by plugging them into $f(x)$) we see that 1 is a root. Then, by dividing $f(x)$ by $(x-1)$ we get

$$f(x)=(x-1)(x^3-3x^2+4x-2)$$

So, to get the other roots we need to solve the equation $x^3-3x^2+4x-2=0$. Using the rational root theorem again we get $-2,-1,1$ and 2 as the possible rational roots. By plugging these values in, we see that 1 is a root of the cubic factor also. Therefore we see that 1 is a multiple root of $f(x)$. As we did before, we use synthetic division on the cubic factor to factor this even more. We get

$$x^3-3x^2+4x-2=(x-1)(x^2-2x+2)$$

Since the quadratic term factors have the roots $-1+i$ and $-1-i$. It follows that the roots of $f(x)$ are $-1+i$ and $-1-i$, and 1 (double root).

4.2 Conjugate Root Theorems

From the worked out examples in the previous section, we are able to see some patterns about the roots. For instance, we saw that both $1+i$ and $1-i$ were both roots of the same polynomial. Also, a different polynomial had both $1+\sqrt{2}$ and $1-\sqrt{2}$ as roots. The following theorems will justify this behavior of non-real and irrational roots.

Theorem 4.3 (Complex Conjugate Root Theorem). *If a polynomial with real coefficients has* $a+bi$ *as a root, then* $a-bi$ *is also a root of the same polynomial.*

Theorem 4.4 (Irrational Conjugate Root Theorem). *If a polynomial with rational coefficients has* $a+b\sqrt{c}$ *as a root, then* $a-b\sqrt{c}$ *is also a root of the same polynomial.*

Remark 4.1. The theorems above are saying that, given the necessary hypothesis, non-real (irrational) roots always come in pairs.

Conjugate Root Theorems: Worked out examples

1. *A polynomial with rational coefficients has $2, 2+\sqrt{3}$ and $1+5i$ as roots. What is the minimum degree that this polynomial can have?*
   Answer: Note that, because irrational roots of the type given and complex roots come in pairs, having $2+\sqrt{3}$ and $1+5i$ as roots force $2-\sqrt{3}$ and $1-5i$ to also be roots of this polynomial.
   So, the polynomial, at the very least, must have the $2, 2+\sqrt{3}, 2-\sqrt{3}, 1-5i$, and $1+5i$ as roots. Therefore, the minimum degree of the polynomial is 5.
2. *The polynomial $f(x) = x^3 - 3x^2 - 2x + c$ has $2+\sqrt{2}$ as a root. Find the value of c and the other roots of $f(x)$.*
   Answer: This question seems to be completely different from others seen before. But do not worry, using fundamentals and the theorems in this section you can readily get the answer.
   We know that the polynomial has $2+\sqrt{2}$ as a root. Then, if we plug this into $f(x)$ we should get zero. On the other hand,
   $$f(2+\sqrt{2}) = (2+\sqrt{2})^3 - 3(2+\sqrt{2})^2 - 2(2+\sqrt{2}) + c = -2 + c$$
   forces $c = 2$.
   Now we know the polynomial is $f(x) = x^3 - 3x^2 - 2x + 2$ and that $2+\sqrt{2}$ is a root. The irrational conjugate root theorem implies that $2-\sqrt{2}$ is also a root of $f(x)$.
   In order to find the other root we summon the rational root theorem. It gives us $-2, -1, 1$ and 2 as the possible rational roots of $f(x)$. Plugging each of these candidates into $f(x)$, we get that -1 is a root of $f(x)$. It follows that the roots of $f(x)$ are $2+\sqrt{2}$, $2-\sqrt{2}$, and -1. Since this polynomial has degree three, these must be the only roots of it.

4.3 Graphs of Polynomials

In this section we will learn how to sketch the graph of a polynomial and also to, given the graph, get information about the polynomial associated with the graph

Steps for sketching graphs of polynomials:
Step 1: Find the roots of the polynomial and mark these points on the x-axis. These are the places where the graph 'meets' the x-axis.
Step 2: If there are multiple roots, check their behavior at the intercepts (where the x-axis meets the graph) using the multiple root test below.

<u>Multiple root test</u>

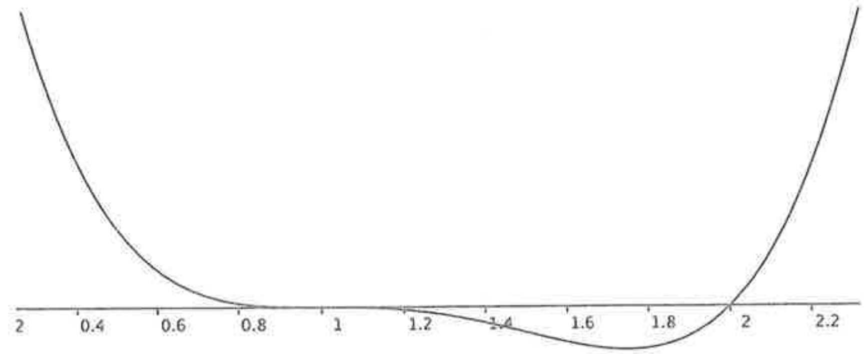
If a polynomial $f(x)$ has a multiple root at $x = c$ of multiplicity n. That is, $(x-c)^n$ is a factor of $f(x)$ but $(x-c)^{n+1}$ is not a factor of $f(x)$.

(a) If c is of even multiplicity, then the graph touches the x axis at $x = c$ but does not cross it.

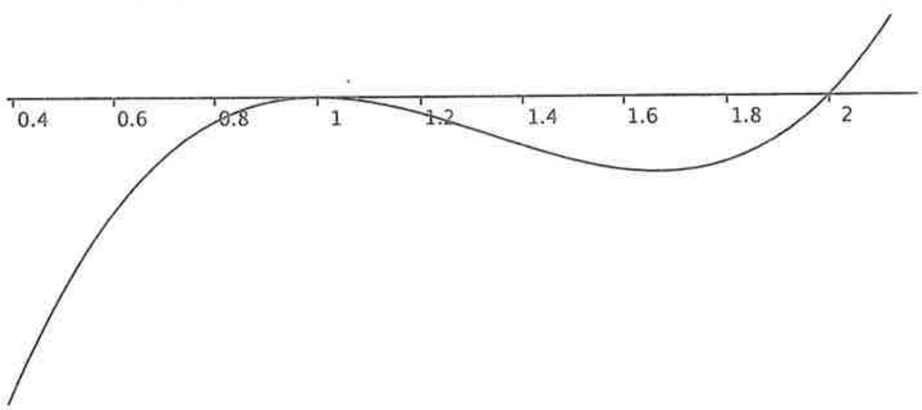
(b) If c is of odd multiplicity, then the graph crosses the x-axis at $x = c$.

The multiple root test is illustrated in the following examples.

Example 4.3. The polynomial $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2)$ has a root of multiplicity 3 at $x = 1$. So, the graph crosses the x-axis at $x = 1$. See graph below



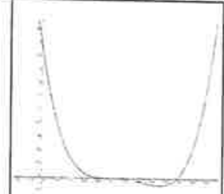
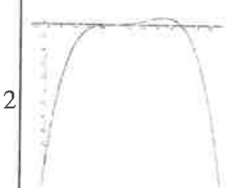
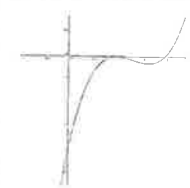
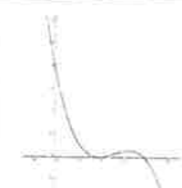
Example 4.4. The polynomial $f(x) = x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2)$ has a root of multiplicity 2 at $x = 1$. So, the graph hits the x-axis and bounces back at $x = 1$. See graph below



After doing steps 1 and 2 you proceed to
Step 3: Find the end behavior of the function using the leading term test:
By 'end behavior' we mean how the graph behaves at the far left and far right ends of the x-axis (as x approaches negative infinity or positive infinity).
The end behavior test, which will help you to decide the end behavior of a polynomial just by checking the degree of the polynomial and its leading coefficient, is presented as a table in the following page.
Step 4: Find the y-intercept of the graph (i.e. the value of y when $x = 0$).

Using the preceding four steps we can sketch the graph of many polynomials. As a general 'recipe': First draw the end behavior (step 3), then look at the roots and whether the graph crosses the axis at those points (step 2), then mark the y-intercept (step 3) and draw the sketch accordingly.

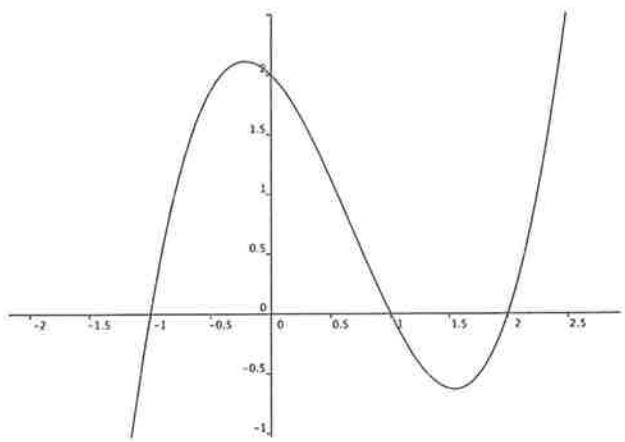
The end behavior test

Type	Degree	Leading Coefficient	Behavior of the graph at the left of the x-axis	Behavior of the graph at the right of the x-axis	Example	Graph
A	even	positive	way above the x-axis	way above the x-axis	$f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$	
B	even	negative	way below the x-axis	way below the x-axis	$f(x) = -x^4 + 5x^3 - 9x^2 + 7x - 2$	
C	odd	positive	way below the x-axis	way above the x-axis	$f(x) = x^3 - 4x^2 + 5x - 2$	
D	odd	negative	way above the x-axis	way below the x-axis	$f(x) - x^3 + 4x^2 - 5x + 2$	

Graphs of Polynomials: Worked out examples

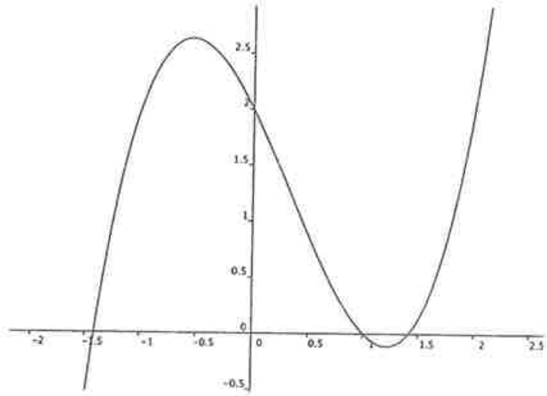
1. *Sketch the graph of* $f(x) = x^3 - 2x^2 - x + 2$.
   Answer: Let us go through each of steps discussed above.
   Step 1 (roots, x-intercepts): In section 4.1, worked out problem 1 we found the roots of this polynomial to be $-1, 1$ and 2. Hence, we can write $f(x) = (x-1)(x+1)(x-2)$.
   Step 2 (multiple roots): Since there are no multiple roots, the graph crosses the x-axis at each of the roots $x = -1, x = 1$ and $x = 2$.
   Step 3 (end behavior): Since the degree of $f(x)$ is 3, which is odd, and the leading coefficient is 1, which is positive, then this graph is of type C. So, the graph will start (on the left) way below the x-axis and will end up (on the right) way above the x-axis.
   Step 4 (y-intercept): When $x = 0$, we get $y = 2$.

   Because this is a type C graph (step 3), we have a graph which starts way below the x-axis at the left end and ends way above the x-axis at the right end. We also know (step 2) that the three roots produce three points where the graph crosses the x-axis. If you mark these points on the x-axis and try to create a curve that goes through them, you will see that the graph comes (from left to right) from way below the axis, then crosses the x-axis at $x = -1$, then crosses the y-axis and turns back down so it can cross the axis at $x = 1$. Finally, the graph must turn back up, cross the x-axis again at $x = 2$ and end way above the x-axis. We are almost done! The only piece of information missing is where the graph crosses the y-axis, information obtained in step 4. The graph will look like the following:



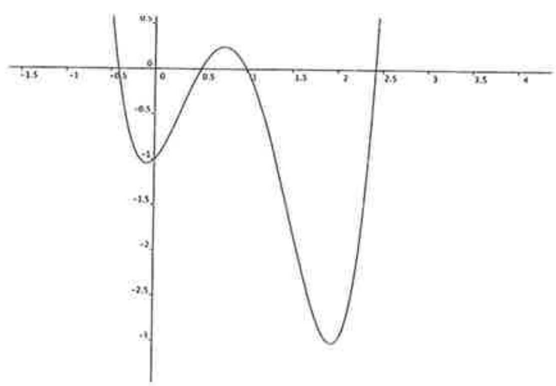
2. *Sketch the graph of* $f(x) = x^3 - x^2 - 2x + 2$.
   Answer: Follow the four steps explained above.
   Step 1 (roots, x-intercepts): In section 4.1, worked out problem 2, we found that the three roots of the polynomial are $1, \sqrt{2}$, and $-\sqrt{2}$. The polynomial can be factored as $f(x) = (x-1)(x+\sqrt{2})(x-\sqrt{2})$.
   Step 2 (multiple roots): Since there are no multiple roots, the graph crosses the x-axis at each of the roots $x = 1, x = \sqrt{2}$, and $x = -\sqrt{2}$.
   Step 3 (end behavior): Since the degree of $f(x)$ is 3, which is odd, and the leading coefficient is 1, which is positive, then this graph is of type C.
   Step 4 (y-intercept): When $x = 0$, we get $y = 2$.

   This graph is very similar to the one studied in the previous exercise; the end behavior is the same, same number of roots, no multiple roots, same y-intercept, etc. The only difference will be where this graph will cross the x-axis. Keeping this in mind, we get that the graph will look like the following



3. *Sketch the graph of* $f(x) = 2x^4 - 7x^3 + 5x^2 + x - 1$.
Answer: Again, we will follow the four steps previously used.
Step 1 (roots, x-intercepts): In section 4.1, worked out problem 5 we found that the three roots of the polynomial are $1,\ \frac{1}{2},\ 1+\sqrt{2},\ 1-\sqrt{2}$.
Step 2 (multiple roots): Since there are no multiple roots, the graph crosses the x-axis at each of the roots $x=1,\ x=\frac{1}{2},\ x=1+\sqrt{2},\ x=1-\sqrt{2}$.
Step 3 (end behavior): Since the degree of $f(x)$ is 4, which is even, and the leading coefficient is 1, which is positive, then this graph is of type A. That is, both at the far left and far right, the graph is way above the x-axis.
Step 4 (y-intercept): When $x=0$, we get $y=-1$.

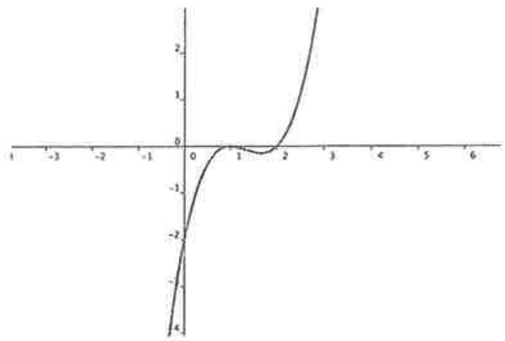
Using the same reading of the data, obtained in the four steps used in previous problems, we get that the graph starts way above the x-axis at the left end and then continues decreasing until it crosses the x-axis at $x=1-\sqrt{2}$, then it turns back up, crossing the y-axis at $y=-1$ to then cross the x-axis at $x=\frac{1}{2}$. The graph then must go turn back down and cross the x-axis at $x=1$, then turn back up, cross the x-axis, for a last time at $x=1+\sqrt{2}$ and then continue increasing to finish way above the x-axis. The graph will look like the following:



4. *Sketch the graph of* $f(x) = x^3 - 4x^2 + 5x - 2$.
Answer: We will proceed as usual, doing the four steps for graphing polynomials.
Step 1 (roots, x-intercepts): By using the methods learned in previous sections we find the three roots of the polynomial. They are $1, 1$ and 2. Hence, $f(x) = (x-1)^2(x-2)$. Notice that 1 is a multiple root.
Step 2 (multiple roots): Since 1 is a multiple root of multiplicity two, then the graph must touch the x-axis and turn back at $x=1$. Since $x=2$ is not a multiple root, then the graph will cross the x-axis at $x=2$.
Step 3 (end behavior): Since the degree of $f(x)$ is 3, which is odd, and the leading term is 1, which is positive, then this graph is of type C. So, the graph will start way below the x-axis at the left and will end up way above the x-axis on the right.

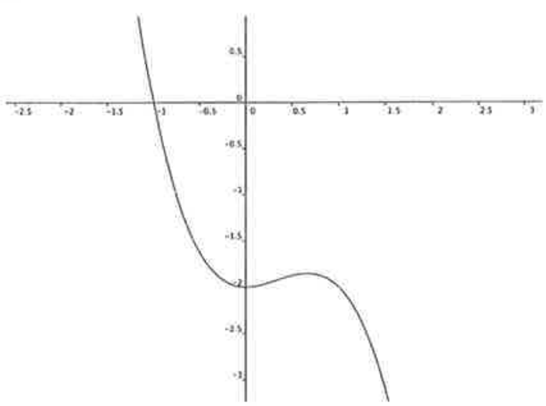
Step 4 (y-intercept): When $x = 0$, we get $y = -2$.

Now we read the information we have just obtained. We see that this graph will start way below the x-axis at the left end and then go up until crossing the y-axis (as both of the x-intercepts are positive) at $y = -2$. Then, the graph will go towards the x-intercept $x = 1$. At that point the graph will turn back and start descending again (without crossing the x-axis). Since there is a second x-intercept at $x = 2$ then the graph must turn back up and cross the x-axis at $x = 2$. After this the graph keeps going up and ends up way above the x-axis. The graph will look like what is given below.



5. *Sketch the graph of* $f(x) = -x^3 + x^2 - 2$.
Answer: The four steps for graphing polynomials follow:
Step 1 (roots, x-intercepts): By using the methods learned in previous sections, we find the three roots of the polynomial. They are $1, 1+i$ and $1-i$. Note that since there is only one real root, even if there are three roots, the graph will cut the x-axis at only one place.
Step 2 (multiple roots): Since there are no multiple roots, the graph crosses the x-axis at $x = -1$. The roots $1+i$ and $1-i$ do not yield any x-intercepts as the x-axis consists of only real numbers.
Step 3 (end behavior): Since the degree of $f(x)$ is 3, which is odd, and the leading term is -1, which is negative, this graph is of type D. So, the graph will start way above the x-axis at the left and will end up way below the x-axis on the right.
Step 4 (y-intercept): When $x = 0$, we get $y = -2$.

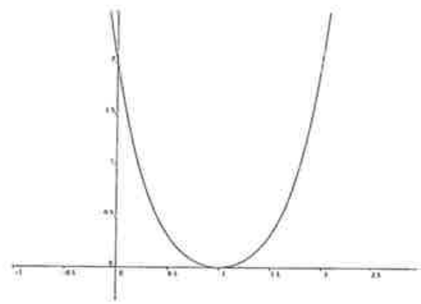
Since this graph is of type D, we have a graph which starts way above the x-axis at the left and then start decreasing. Eventually this graph will cross the x-axis at $x = -1$ and then cross the y-axis at $y = -2$. At this point we have run out of reference points (intercepts), so we look at the end behavior to the right, and since it says that the graph ends way below the x-axis, the graph will keep going down forever. The graph will look like what is given below.



6. *Sketch the graph of* $f(x) = x^4 - 4x^3 + 7x^2 - 6x + 2$.
Answer: We will use the four steps for graphing, just as we have done before.
Step 1 (roots, x-intercepts): In section 4.1, worked out problem 6, we found that the three roots of the

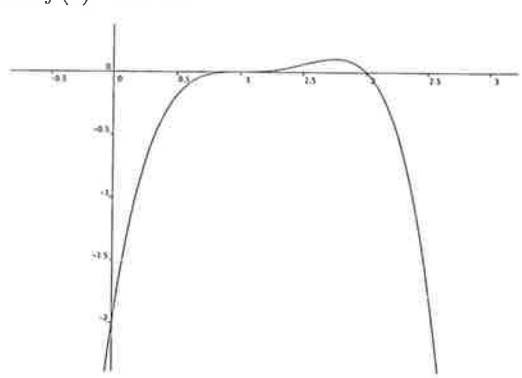
polynomial are 1, 1 (double root), $-1+i$, and $-1-i$.
Step 2 (multiple roots): Since there is a multiple root at $x=1$ of multiplicity 2, the graph will touch the x-axis at $x=1$ and turn back. The remaining two roots are complex roots so they will not appear in the graph as x-intercepts.
Step 3 (end behavior): Since the degree of $f(x)$ is 4, which is even, and the leading term is 1, which is positive, this graph is of type A. So, the graph will start way above the x-axis at the left and will also end up way above the x-axis on the right.
Step 4 (y-intercept): When $x=0$, we get $y=2$.

Since this is a type A graph, it will start way above the x-axis on the left. After that, the graph will start decreasing until crossing the y-axis at $y=2$, then it will continue decreasing until hitting the x-axis at $x=1$, which is a double root, where it will turn back up without crossing the x-axis. Since there are no more roots and the graph must end way above the x-axis, the graph will keep going up forever. The graph should look like the following:

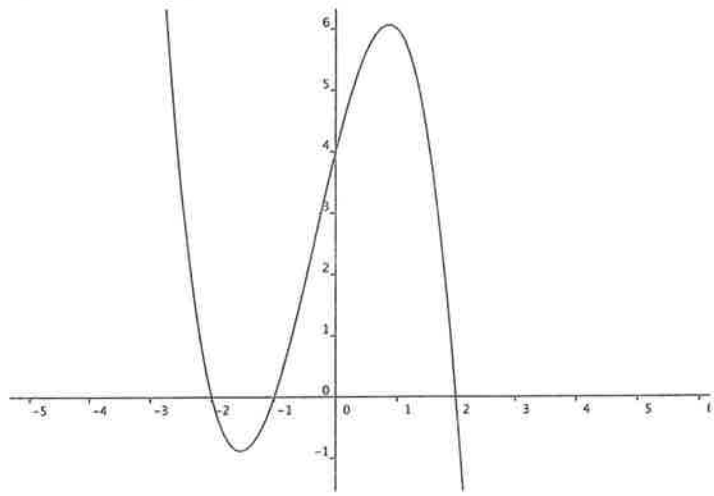


7. *Sketch the graph of* $f(x)=-x^4+5x^3-9x^2+7x-2$.
Answer: We will use the four steps for graphing, just as we have done before.
Step 1 (roots, x-intercepts): By using the methods learned in previous sections, we find the four roots of the polynomial. They are $1,1,1$ and 2, where 1 is a root of multiplicity three.
Step 2 (multiple roots): Since there is a multiple root at $x=1$ of multiplicity 3, which is odd, the graph will cross the x-axis at $x=1$. The remaining root, $x=2$, is not a multiple root so the graph will also cross the x-axis at this point.
Step 3 (end behavior): Since the degree of $f(x)$ is 4, which is even, and the leading term is -1, which is negative, this graph is of type B. So, the graph will start way below the x-axis at the left and will also end up way below the x-axis on the right.
Step 4 (y-intercept): When $x=0$, we get $y=-2$.

Since this is a type B graph, the graph will start way below the x-axis on the left and then it will go upwards, it will cross the y-axis at $y=-2$ and then cross the axis at $x=1$. After that it will have to turn back down so it can cross the x-axis again at $x=2$. The graph will continue descending until it ends up way below the x-axis on the right. The graph of $f(x)$ follows:



8. *Which of the following must be true about the graph of a polynomial given below*



a. *The polynomial has at least one complex root*
b. *The polynomial has $(x+1)$ as a factor*
c. *The polynomial is divisible by $(x+1)(x^3-3x^2+4x-2)$*

Answer: As you can see from the graph, it cuts the x-axis at three different points: $-2, -1$, and 2, and these are 3 real roots which give factors $(x+2)$, $(x+1)$ and $(x-2)$. This is as much as we can say. That is, it is possible that some of these roots have multiplicity larger than one (always odd, though, because the graph crosses the x-axis at these points), and it is also possible that the polynomial has some complex roots, but we cannot be sure of these facts. In other words, all these claims do not *have to* happen. It follows that b. is the only option that *must* occur.

4.4 The Sum and Product of the Roots of a Polynomial

In this section we will discuss the relationship between the values of roots and coefficients of a polynomial and how they can be used to solve problems.

Consider the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, then

(a) The sum of the roots of $f(x)$ is equal to $-\dfrac{a_{n-1}}{a_n}$

(b) The product of the roots of $f(x)$ is equal to $(-1)^n \dfrac{a_0}{a_n}$

We will use an example to explain this. Assume that a, b and c are the three roots of a cubic polynomial. Using the factor theorem, we know that $f(x) = a_3(x-a)(x-b)(x-c)$, where a_3 is the leading coefficient of $f(x)$. Multiplying the right-hand side expression, we get

$$f(x) = a_3x^3 - a_3(a+b+c)x^2 + a_3(ab+ac+bc)x - a_3abc$$

It follows that the coefficient of the second leading term gives the sum of the roots times a_3, and that the constant term gives the product of the roots times a_3. Dividing both expressions by a_3 yields exactly what our formulas above say.

Example 4.5. Assume we know that one of the roots of the polynomial $f(x) = x^3 - 11x + c$ is $2+i$. We want to find the value of c and the other roots of $f(x)$.

This problem is similar to worked out problem 2 in section 4.2, and thus it could be solved in the same way. However, we want to solve this problem using the results obtained in this section.

Since $2+i$ is one of the roots, the complex conjugate root theorem forces $2-i$ to be a root of $f(x)$ as well. Let γ be the third root of $f(x)$. Since the polynomial has degree 3, these are the only roots. Now we use what we have learned in this section to get

$$(2-i)+(2+i)+\gamma=\frac{-0}{1}=0 \qquad (2-i)(2+i)\gamma=(-1)^3\frac{c}{1}=-c$$

The first equation can be re-written as $4+\gamma=0$, and so $\gamma=-4$. Plugging this into the second equation, we get $(2-i)(2+i)(-4)=-c$. Multiplying all that out, we get $c=20$.

4.5 Rational Functions

We take inspiration from the definition of a rational number; a fraction of two integers to define a rational function as a fraction of two polynomial functions.

An example of a rational function is

$$R(x)=\frac{2x^3+4x^2+5}{3x^2+5x+6}$$

Notice that both the numerator and the denominator of $R(x)$ are polynomials.

If a rational function can be written as $R(x)=\frac{f(x)}{g(x)}$ then the domain of $R(x)$ is defined to be the set of all values x such that $g(x)\neq 0$. In other words, it is the set of values of x where the polynomial in the denominator of $R(x)$ is not zero. For example, the domain of the rational function $R(x)=\frac{x^2+x+1}{3x-6}$ is the set of real numbers other than 2, as $x=2$ is the only root of the denominator polynomial. Since the function is not defined at 2, the graph of the function will have a *discontinuity* at 2.

Remark 4.2. According to what we have learned the domain of the function

$$f(x)=\frac{x^2-2x+1}{x^2-1}$$

is all real numbers except for $x=\pm 1$, because these values make the denominator equal to zero. However, if we factor the numerator and denominator of the expression given for $f(x)$ we get

$$\frac{(x-1)^{\not 2}}{(x+1)\cancel{(x-1)}}=\frac{x-1}{x+1}$$

So, it seems that $x=1$ is a point of the domain of $f(x)$. However, what happens here is that even though the (algebraic) simplification is correct, it does transform the original function $f(x)$ into a *different* function

$$g(x)=\frac{x-1}{x+1}$$

Hence, the domain of $f(x)$ is still all real numbers expect for ± 1, and the domain of $g(x)$ is all real numbers different from $x=-1$. These two functions agree in all values of x, except at $x=1$, where $f(x)$ is undefined and $g(1)=0$.

Domain and Intercepts

Example 4.6. Let us find the domain and the x and y-intercepts of the rational function $f(x)=\frac{x^2-x-6}{x^2-6x+8}$.

Domain: In order to find the domain, we find the roots of the polynomial in the denominator of the function by setting it to zero and solving for x. Then we define the domain as all real numbers different from these roots. Hence, we want ti solve

$$x^2 - 6x + 8 = 0$$

It follows that, by factoring,

$$(x-4)(x-2) = 0$$

and thus $x = 2$ or $x = 4$.

We get that the domain of $f(x)$ is the set of all real numbers except 2 and 4.

x-intercept: We do what we usually do when we want to find the x-intercept of a function, which is simply setting the function equal to zero and solve for x. In this case, after we set $\frac{x^2 - x - 6}{x^2 - 6x + 8} = 0$ we can multiply both sides by the denominator, this yields the equation obtained by setting the numerator of $f(x)$ equal to zero. Then we just need to solve for x:

$$x^2 - x - 6 = 0$$

After factoring we get

$$(x-3)(x+2) = 0$$

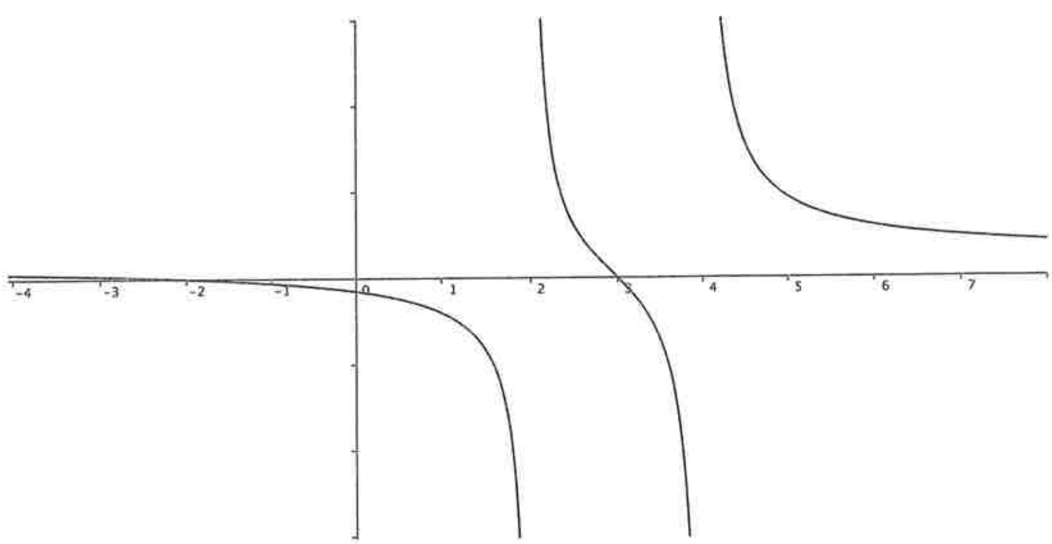
and thus, the x-intercepts of $f(x)$ are $x = 3$ and $x = -2$.

y-intercept: Then again, there is nothing new here. You may find the y-intercept of $f(x)$ by simply substituting $x = 0$ and the finding the appropriate value for y. Hence, we get

$$f(0) = \frac{-6}{8} = -\frac{3}{4}$$

Therefore, the y-intercept of $f(x)$ is $y = -\frac{3}{4}$.

The graph of $f(x)$ follows, there you can clearly see the discontinuities in the graph at 2 and 4 (the points not in the domain) and the intercepts.

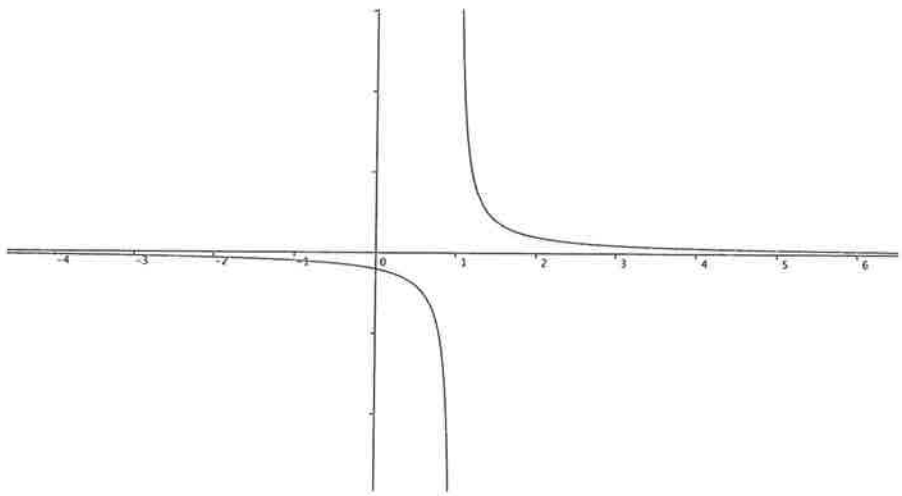


Later in this section we will show you how to graph a rational function. Graphs like the one above are very common for rational functions.

Vertical and Horizontal Asymptotes

If you take a good look at the graph above you will see that near the points of discontinuity ($x = 2$ and $x = 4$) the values of $f(x)$ become gigantically large, both in the positive direction (up) and the negative direction (down), all this depending on the direction from which you approach these discontinuities. These are telltale signs of an object known as vertical asymptotes. Let us take a simpler function to study it more closely.

Let us consider the rational function $f(x) = \dfrac{1}{x-1}$. Since the denominator $x-1$ has a root at $x = 1$, then $x = 1$ is not in the domain of $f(x)$, and thus the function has a discontinuity at $x = 1$. Let us now see the graph of this function to see the behavior near $x = 1$ (note that we always use the word 'near' as we cannot really 'touch' $x = 1$ because $f(x)$ is not defined at $x = 1$).



You will notice that as x goes near 1 from the left, the values for $f(x)$ decrease towards $-\infty$, and as x goes near 1 from the right the values fro $f(x)$ increase towards ∞. When this type of behavior happens, we say that the vertical line at the discontinuity (in this case $x = 1$) is a vertical asymptote.

Remark 4.3. In mathematical notation we write

$$\lim_{x \to 1^+} f(x) = \infty \qquad\qquad \lim_{x \to 1^-} f(x) = -\infty$$

or

$$f(x) \to \infty \text{ as } x \to 1^+ \qquad\qquad f(x) \to -\infty \text{ as } x \to 1^-$$

to mean that the values of $f(x)$ grow towards infinity when x approaches 1 by the right and that the values of $f(x)$ decrease towards $-\infty$ when x approaches 1 by the left.

The symbols $\to 1^-$ and $\to 1^+$ mean that 1 is approached from the left/right (respectively). Be careful not confusing 1^- and 1^+ with -1 and $+1$!

Now that we have seen an example of a what a vertical asymptote is it is time to give a more formal definition for these objects.

Definition 4.1 (Vertical Asymptote). A vertical line $x = c$ is said to be a vertical asymptote to a function $f(x)$ if the function goes to $\pm\infty$ as x approaches c from the left or from the right. That is, $x = c$ is a vertical asymptote of $f(x)$ if and only if

$$f(x) \to \pm\infty \text{ as } x \to 1^{\pm}s$$

This definition is not only valid for rational functions. However, the concept of vertical asymptote is easy to work with in the case that $f(x)$ is a rational function.

Theorem 4.5. *Let $f(x)$ be a rational function. If $x = c$ is a root of the denominator of $f(x)$ and is NOT a root of the numerator, then $x = c$ is a vertical asymptote of $f(x)$.*

Remark 4.4. Not all vertical asymptotes can be found using Theorem 4.5 but most of them can. In particular, all vertical asymptotes in this book may be found this way.

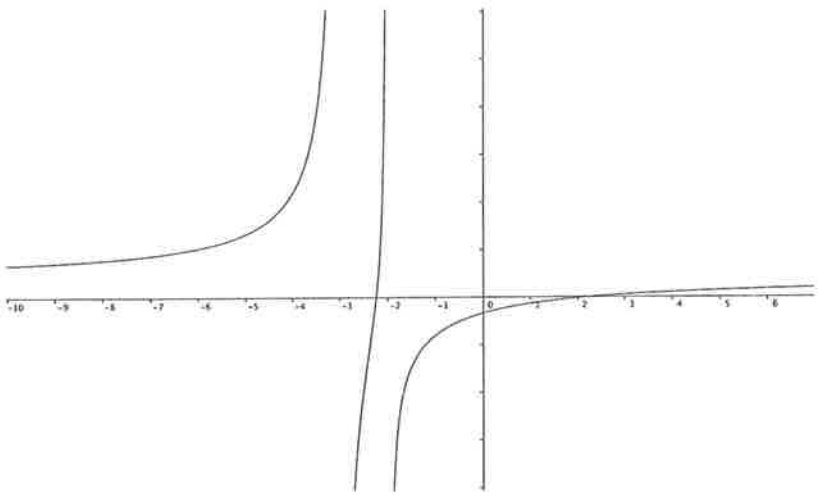
Example 4.7. We want to find the vertical asymptotes of the rational function

$$f(x) = \frac{2x^2 - 10}{x^2 + 5x + 6}$$

We first look for the roots of the denominator.

$$0 = x^2 + 5x + 6 = (x+3)(x+2)$$

and thus, we get $x = -2$ and $x = -3$. Since none of these two values is a root of the numerator then $f(x)$ has two vertical asymptotes: $x = -2$ and $x = -3$.
This can be verified by looking at the graph of this function below.



If you look closely at the graph above, in addition to the vertical asymptotes, you will also see another aspect of this graph that is interesting; looking at it sidewise you will see a vertical asymptote. This is a *horizontal* asymptote!
Recall that when we graphed polynomials in Section 4.3 we learned that on the far left or the far right of the graph of a polynomial (as $x \to \infty$ and $x \to -\infty$) the graph would only increase or decrease indefinitely (go to $\pm\infty$). For rational functions this behavior is also possible, but it also very common to you get horizontal lines that are approached (on the far left/right) by the graph, but never touched. These lines are called horizontal asymptotes.

Definition 4.2 (Horizontal Asymptote). A horizontal line $y = b$ is said to be a horizontal asymptote to the graph of $f(x)$ if the function approaches the value $y = b$ as x approaches $\pm\infty$. This, in mathematical notation, is written

$$\lim_{x \to \pm\infty} f(x) = b \qquad \text{or} \qquad f(x) \to b \text{ as } x \to \pm\infty$$

Once again, the definition above is a general definition valid for all functions. We have a user-friendly theorem that may be applied to find the horizontal asymptote(s) of a rational function.

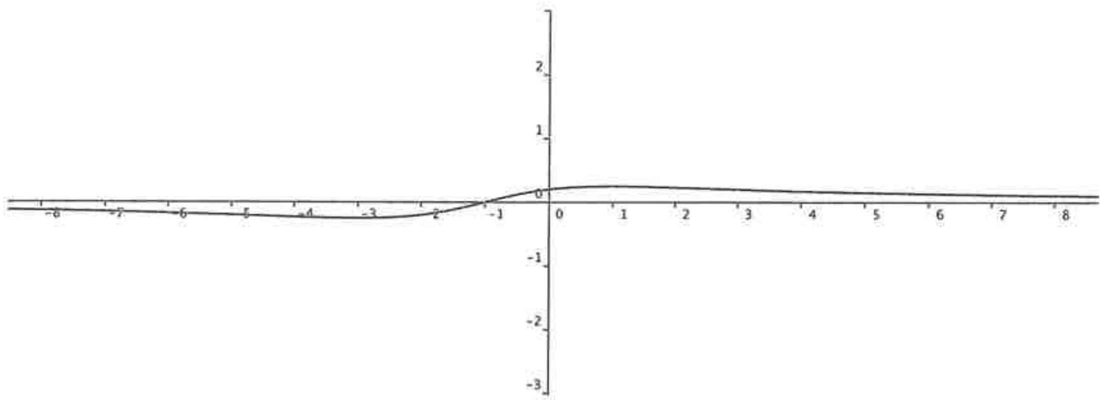
Theorem 4.6. *Let $R(x) = \dfrac{f(x)}{g(x)}$ be a rational function. Then,*

1. *if the degree of the denominator is greater than the degree of the numerator, then the x-axis (the line $y = 0$) is a horizontal asymptote to the graph of $R(x)$.*
2. *if the degree of the denominator is less than the degree of the numerator, then the graph of $R(x)$ has no asymptotes and the function goes to $\pm\infty$ at the left/right ends.*
3. *if the degree of the denominator is equal to the degree of the numerator, then the line $y = \frac{a}{b}$ is a horizontal asymptote, where*
   a = leading coefficient of the numerator.
   b = leading coefficient of the denominator.

Example 4.8. Let us find the horizontal asymptotes (if they exist) for the following functions

$$f(x) = \frac{x+1}{x^2+2x+5} \qquad g(x) = \frac{2x^2-5}{3+2x+6x^2} \qquad h(x) = \frac{x^3+x+1}{2x+1}$$

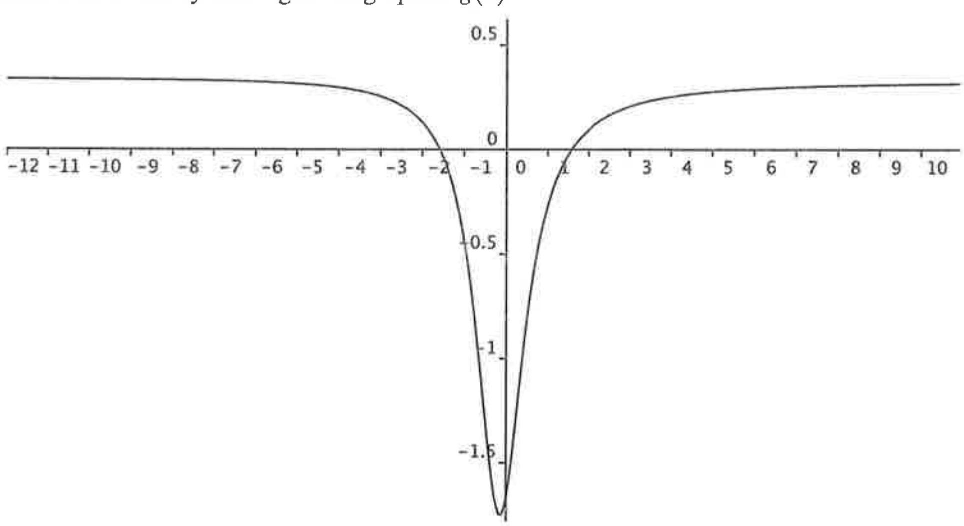
For $f(x)$, the degree of the denominator is 2, which is greater than the degree of the numerator (which is 1). Therefore, Theorem 4.6 says that there is one horizontal asymptote; the x-axis. You can see this clearly in the graph below.



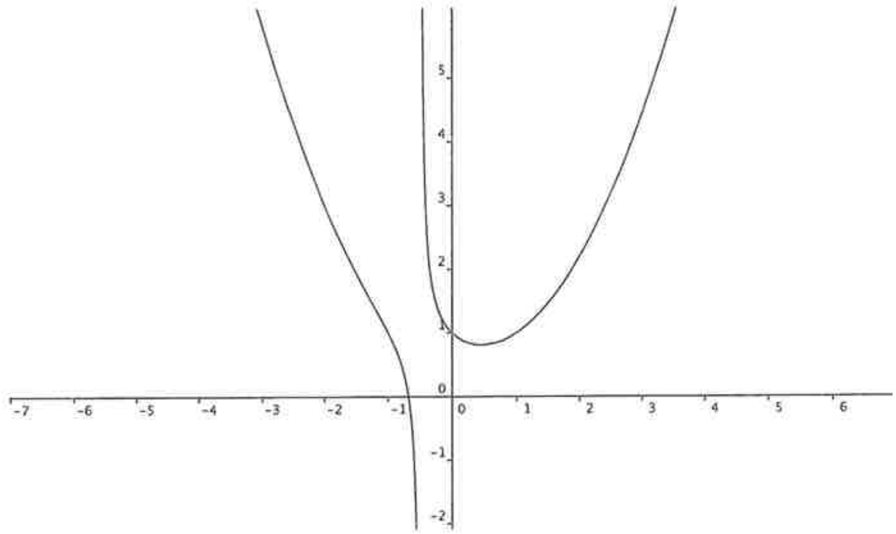
For $g(x)$, the degrees of both the numerator are equal to 2. So, this rational function has a horizontal asymptote given by the leading coefficients of the numerator and denominator. In fact, since the leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 6, then the horizontal asymptote of $g(x)$ is

$$y = \frac{2}{6} = \frac{1}{3}$$

You can check this is true by looking at the graph of $g(x)$:



For $h(x)$, we see that the degree of the denominator is 1, which is smaller than the degree of the numerator (which is 3). Therefore, Theorem 4.6 saysthat $h(x)$ has no horizontal asymptotes. This may be observed in the graph of $h(x)$.



Sketching the Graph of a Rational Function

Let us now consider the art of sketching the graph of rational functions. This is a matter of bringing together the attributes which are important to rational functions such as intercepts, vertical asymptotes and horizontal asymptotes together in a coherent manner to produce a graph. We will outline this process using four steps.
Step 1: Find any x-intercepts by setting the numerator to zero and solving for x, and the y-intercept by plugging $x = 0$ into the function. Plot all intercepts in the coordinate axes.
Step 2: Find any vertical asymptotes using Theorem 4.5. Draw these vertical lines as dashed/dotted lines.
Step 3: Find any horizontal asymptote by using Theorem 4.6. Draw these vertical lines as dashed/dotted lines.
Step 4: Complete the graph by finding a few additional points, at least one for each interval in the domain (those separated by vertical asymptotes). Make sure that the graph does not touch the vertical asymptotes at all. The graph should not touch the horizontal asymptotes at the ends but it is possible that the graph may cut a horizontal asymptote in the middle area of the graph.

Example 4.9. We want to sketch the graph of $f(x) = \frac{2x-6}{x-2}$. We will go through the four steps described above.
Step 1: Finding intercepts. We set the numerator equal to zero, we get

$$2x - 6 = 0$$

and thus $x = 3$. It follows that there is exactly one x-intercept: $x = 3$.
Now we plug $x = 0$ into $f(x)$ to get

$$f(0) = \frac{-6}{-2} = 3$$

Hence, the y-intercept is $y = 3$.
Step 2: Finding vertical asymptotes. We set the denominator equal to zero, we get

$$x - 2 = 0$$

which implies $x = 2$. Since $x = 2$ is not a root of the numerator, then $x = 2$ is the only vertical asymptote of $f(x)$.

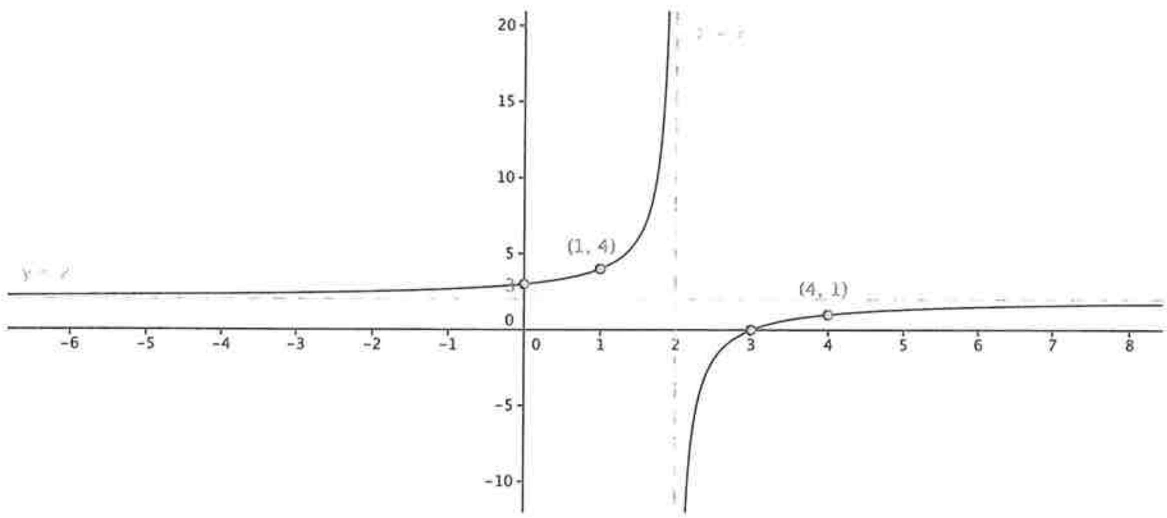
Step 3: Finding horizontal asymptotes. We first check the degrees of the numerator and the denominator. Since both are equal to 1 then Theorem 4.6 says that the horizontal asymptote is given by the leading coefficients of the numerator and denominator. We get that the horizontal asymptote has equation

$$y = \frac{2}{1} = 2$$

Step 4: Finding additional points and sketching the graph. We will find one additional point in each of the intervals in separated by the vertical asymptote Hence, we will pick a point on each of the intervals $(-\infty, 2)$ and $(2, \infty)$. For convenience, let us pick $x = 1$ and $x = 4$. Plugging these into $f(x)$ we get

$$f(1) = \frac{2-6}{1-2} = 4 \qquad\qquad f(4) = \frac{8-6}{4-2} = 1$$

So, the points $(1,4)$ and $(4,1)$ are also on the graph of $f(x)$.
Putting all this together, you can obtain the graph of the function, given below



Example 4.10. We want to sketch the graph of the rational function $g(x) = \dfrac{x^2-4x+4}{x^2+x-2}$. Let us go through the four steps.
Step 1: Finding intercepts. By setting the numerator equal to zero we get,

$$x^2 - 4x + 4 = 0$$

which has solution $x = 2$. Hence, there is exactly one x-intercept: $x = 2$.
Plugging $x = 0$ into $g(x)$ we get

$$g(0) = \frac{4}{-2} = -2$$

It follows that the y-intercept is $y = -2$.
Step 2: Finding vertical asymptotes. We set the denominator equal to zero.

$$x^2 + x - 2 = 0$$

to get $x = -2$ and $x = 1$. Since none of these numbers is a root of the numerator, then $g(x)$ has two vertical asymptotes: $x = -2$ and $x = 1$.
Step 3: Finding horizontal asymptotes. Since the degree of the numerator is equal to the degree of the denominator, then we know $g(x)$ has one horizontal asymptote, which is given by the leading terms of the denominator and numerator. Using this we get that

$$y = \frac{1}{1} = 1$$

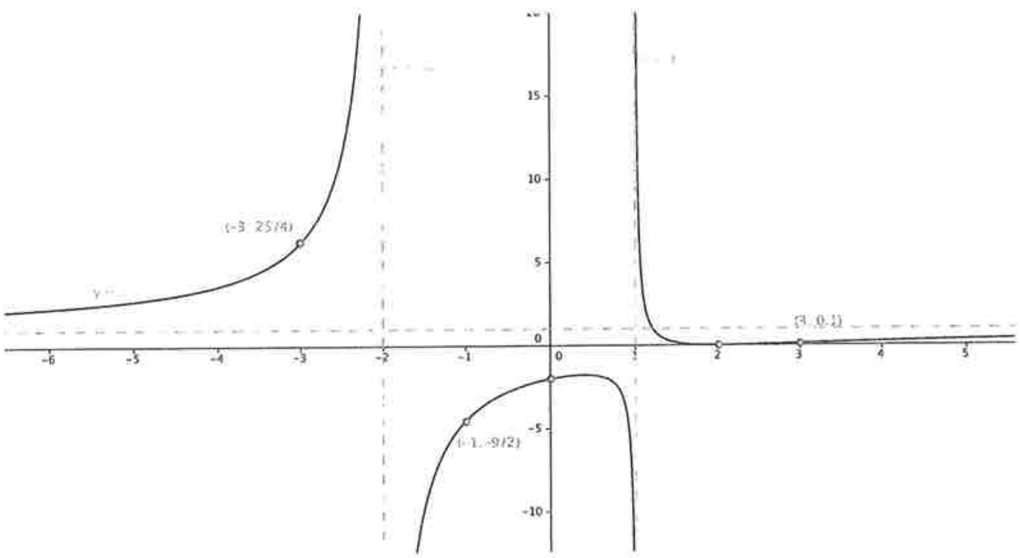
is a horizontal asymptote for $g(x)$.
Draw this asymptote as a dashed/dotted line.
Step 4: Finding additional points and sketching the graph. We will find one additional point in each of the intervals separated by the vertical asymptote. Since the vertical asymptotes are $x = -2$ and $x = 1$ then we need to pick one number in each of the intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. We pick $x = -3$, $x = -1$, and $x = 3$. Plugging these values into $g(x)$ we get

$$g(-3) = \frac{(-3)^2 + 12 + 4}{(-3)^2 - 3 - 2} = \frac{25}{4} \qquad g(-1) = \frac{(-1)^2 + 4 + 4}{(-1)^2 - 1 - 2} = -\frac{9}{2} \qquad g(3) = \frac{3^2 - 12 + 4}{3^2 + 3 - 2} = \frac{1}{10}$$

which yield the following points on the graph:

$$\left(-3, \frac{25}{4}\right) \qquad \left(-1, -\frac{9}{2}\right) \qquad \left(3, \frac{1}{10}\right)$$

Putting all this together you should obtain the graph of the function, which is given below.



Exercises

4.1. Factor the polynomial $f(x) = x^3 - 2x^2 - 5x + 6$ and plot the graph.

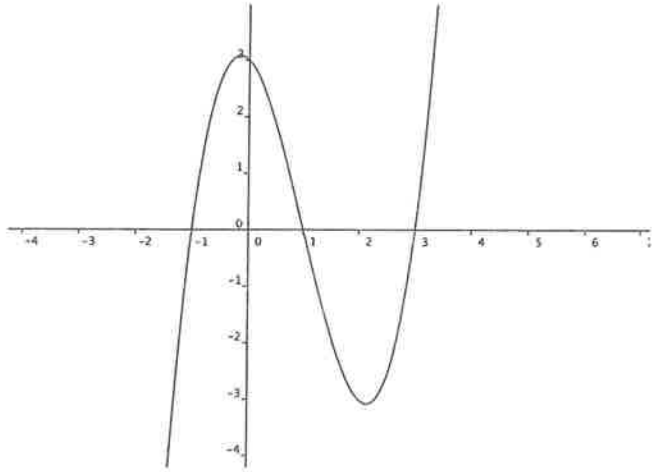
4.2. Factor the polynomial $f(x) = x^3 - 3x^2 - x + 3$ and plot the graph.

4.3. Does the polynomial $f(x) = x^9 + 1$ have any rational roots? Justify your answer.

4.4. Does the polynomial $f(x) = x^9 + 5x^4 + 2$ have any rational roots? Justify your answer.

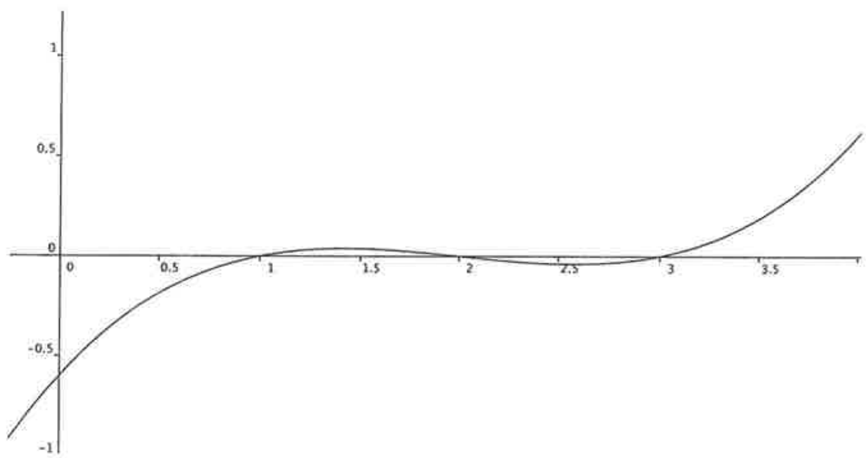
4.5. Find a rational root of $f(x) = 3x^3 - 2x^2 + 3x - 2$, then factor it completely and draw its graph.

4.6. Which of the following statements is true about the following graph



a. The polynomial has at least one complex root.
b. The polynomial has $(x-3)$ as a factor.
c. The polynomial is divisible by x^2+4x+3.

4.7. Which of the following statements is true about the following graph



a. The polynomial has at least one complex root.
b. The polynomial has $(x+1)$ as a factor.
c. The polynomial is divisible by x^2-4x+3.

4.8. Completely factor and draw the graph of $f(x)=2x^3-3x^2-9x+10$.

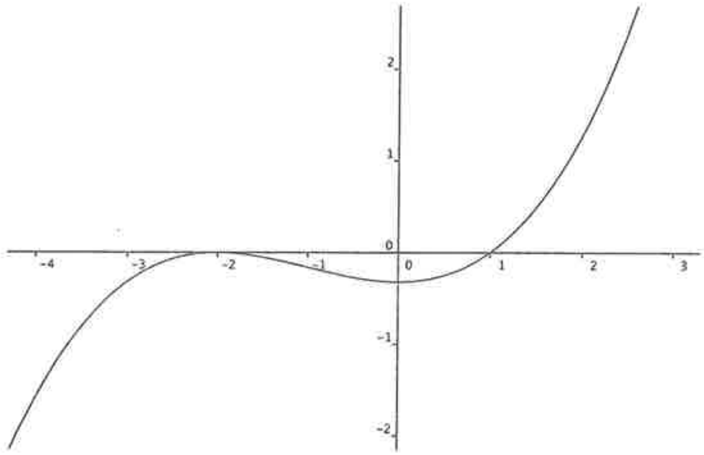
4.9. Completely factor and draw the graph of $f(x)=4x^3+6x^2+2x+3$

4.10. Completely factor and draw the graph of $f(x)=-x^3+5x^2-5x+1$

4.11. Completely factor and draw the graph of $f(x)=x^4+3x^3-11x^2-3x+10$

4.12. Completely factor and draw the graph of $f(x)=x^3-3x^2+4$

4.13. What must be true about the polynomial whose graph is given below?

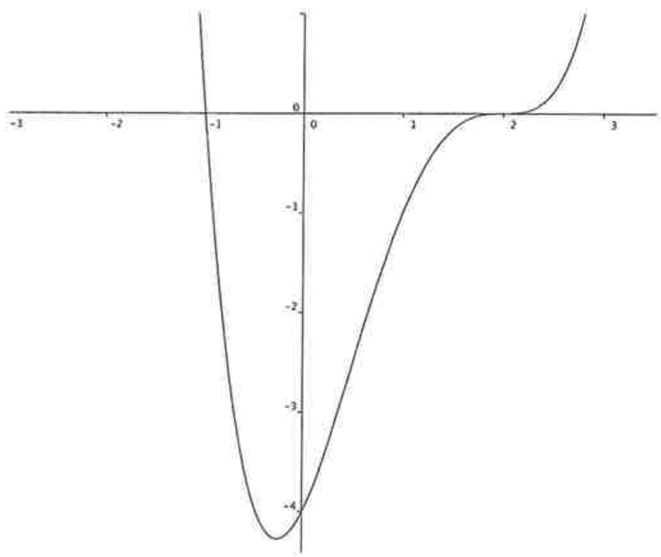


a. The polynomial has no complex roots.
b. The polynomial has a multiple root at $x = -2$.
c. The polynomial must be cubic.

4.14. Completely factor and plot the polynomial $f(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$.

4.15. Completely factor and plot the polynomial $f(x) = x^4 - 2x^3 + 2x - 1$.

4.16. What must be true about the polynomial whose graph is given below?



a. The polynomial must be of degree 2.
b. The polynomial must have non-real roots.
c. The polynomial has no repeated root.
d. None of the above.

4.17. You are given that one of the roots of the polynomial $f(x) = x^3 - x + c$ is $1 + i\sqrt{2}$. Find c, the other roots of $f(x)$, and plot the graph of $f(x)$.

4.18. You are given that one of the roots of the polynomial $f(x) = x^3 - 4x^2 + 8x + c$ is $1 + i\sqrt{3}$. Find c and the other roots of $f(x)$.

4.19. Find the largest value of b such that one root of $f(x) = x^3 + 3x^2 + 5x + 3$ is of the form $-1 + ib$.

4.20. A polynomial $f(x)$ has roots $-3, \sqrt{2}$, and $5i$. What is the smallest degree it can have?

4.21. Is it possible for a polynomial of degree 5 to have 3 imaginary roots and two real roots?

4.22. You are given that two of the roots of $f(x) = x^5 + x^4 + x^3 + x^2 - 12x - 12$ are $\sqrt{3}$ and $-2i$

a. Name other irrational and imaginary roots that $f(x)$ can have.
b. What are the possible integer roots of the equation?
c. Out of these possible roots, how many could $f(x)$ actually have?

4.23. Find all the intercepts of the following functions

$(a)\ f(x) = \dfrac{x-2}{3x+12}$ $(b)\ f(x) = \dfrac{x^2-4}{x^2-9}$ $(c)\ f(x) = \dfrac{x^2+9x+14}{x^2-x-12}$ $(d)\ f(x) = \dfrac{x+4}{x^2+25}$

4.24. Find all the asymptotes of the following functions

$(a)\ f(x) = \dfrac{3x-6}{x+5}$ $(b)\ f(x) = \dfrac{3x-6x^2}{9-x^2}$ $(c)\ f(x) = \dfrac{x^3-2x-35}{x^2-x-6}$

$(d)\ f(x) = \dfrac{x+5}{x^2+6x+9}$ $(e)\ f(x) = \dfrac{16-x^2}{3x^2+24x+45}$ $(f)\ f(x) = \dfrac{x^2+7x+12}{x^2+9x}$

4.25. Sketch the graphs of the following functions:

$(a)\ f(x) = \dfrac{1}{x+5}$ $(b)\ f(x) = \dfrac{x}{x-3}$ $(c)\ f(x) = \dfrac{3x-12}{x+5}$

$(d)\ f(x) = \dfrac{x^2-4}{x^2}$ $(e)\ f(x) = \dfrac{x^2+4}{x^2-1}$ $(f)\ f(x) = \dfrac{1}{x^2-4}$

$(g)\ f(x) = \dfrac{5x}{x^2-9}$ $(h)\ f(x) = \dfrac{x^2+5x+6}{x^2-5x+4}$ $(i)\ f(x) = \dfrac{x}{x^2+9}$

$(j)\ f(x) = \dfrac{x^2-4x+3}{x^2+7x+10}$ $(k)\ f(x) = \dfrac{(x^2+5)(x+6)}{(x^2+1)(x-3)}$ $(l)\ f(x) = \dfrac{(x^2-4)(x^2+7x)}{(x^2-9)^2(x+1)}$